Perspectives of modeling COVID-19 transmission via integral equations

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Abstract
The ongoing COVID-19 pandemic has become a major threat to the entire globe. In order to properly place controlling strategies on each level of transmission, researchers, scientists and mathematicians use different approaches to model it. Compartment models such as SIR, SEIR are the center of attention in many models. General concern on integral equation models in disease transmission is considerably low due to the intuitive temptation of modeling in terms of rate of change of a phenomenon. This study expresses possibilities of modeling COVID-19 context in terms of integrals since accumulation effect can be observed in several influencing factors. Both Volterra and Fredholm integral equations can be used to model this, since these influences can accumulate within constant, variable or fixed intervals. While causative factors which consist of cross-references in different platforms can be modeled by degenerated kernels, difference kernels accommodate causative factors with time delay.

Keywords: *COVID-19, Integral Equations, Kernel, Accumulation and Mathematical Modelling*

1 Introduction

The outbreak of coronavirus has attracted extensive attention of many scientists and mathematicians since March 2020. At the present time, there are many mathematical models used to describe this situation in more general perspective as well as regional perspective [1, 2, 3]. As differential equations are always popular in the applications of disease transmission, here also we could observe many models related to differential equations. Compartment models are the center of attention in many models in which system of ordinary differential equations are used to describe the rate of change in each compartment [4]. Thus, the well-known SIR compartmental model was extended according to their topic of interest by introducing novel compartments such as asymptomatic, reported symptomatic infectious, unreported symptomatic infectious etc[3, 5, 6].

This paper focuses on integral equation approach which is not much familiar among applied mathematicians compared to differential models. Even though differential calculus and integral calculus are two opposite operations tied together by the Fundamental Theorem of Calculus, many mathematical treatments via integral equations are not motivated in applications of disease transmission. The emphasis of this study is to widen the scope by seizing the attention of scientists and mathematicians towards the integral equations in modeling disease transmission.

2 Integral Equation

The general form of an integral equation in $u(x)$ of second kind is defined by

$$u(x) = f(x) + \int_a^{b(x)} K(x,t)u(t)dt \tag{1}$$

Here f and b are functions of independent variable x. The function $K(x,t)$ appears within the integral is called the Kernel, which is the main structural entity in both modeling and solving process of integral equations [7]. The most frequently used integral equations fall under two main categories: those with variable limit (eg. $b(x) = x$) of integration is called as **Volterra integral equation** and those with fixed limit (eg. $b(x) = b$) of integration is called as **Fredholm integral equation**. This variable limit integration or Volterra integral equation is twofold, where it can be described either for variable interval or a constant interval.

The solving process holds upon the characteristics of accumulation which is described by the kernel function. Among different types of kernels, difference kernel is prominent as it could introduce the causative factors with time lag. This is formulated by the difference of the arguments $K(x,t) = K(x-t)$. Difference kernel is easily solved by Laplace transforms when it is applied in Volterra integral equations. The degenerated kernel is another commonly observable kernel type which is the easiest way of expressing cross-references of factors in different platforms. This is formulated by $K(x,t) = \sum_{i=0}^{n} A_i(t)B_i(x)$. According to this structure of degenerated kernel, it is easily solvable by separation of variables when it is applied in Fredholm integral equations.

2.1 Variable Interval Integration

Amongst plenty of mathematical models, Hammerstein Volterra integral equation (2), Lotka's Integral model (3) are famous in formulating time-dependent accumulation scenarios accompanied by the time deferment effect.

$$n(t) = n_0 f(t) + k \int_0^t f(t-\tau) n(\tau) d\tau \tag{2}$$

$$n(t) = k \int_0^t f(t) n(t-\tau) d\tau \tag{3}$$

Jerri in his studies on integral equations, use (2) to describe a population dynamic where the survivability is the governing factor [7]. Sequel to that population model, parameters can be redefined to observe the behaviour in epidemiological situation. Under normal circumstance, there is a continuous addition to infected population through the competency of already infected incidences. $n(\tau_i)$ the incidences in a time τ_i have $t - \tau_i$ age of infection by the time t. Thus at the time t, those who infected at τ_i, have the $f(t - \tau_i)$ competency of spreading the disease.

As any other infectious disease, COVID-19 shows a general competency of spreading, that triggers with being infected irrespective of the day they have encountered it. As a matter of course, those infected and close-contacts are quarantined for a certain period. The applicability of the proposed model is examined by applying fundamental exponential behavior ($f(t) = \lambda e^{-at}$) of competency in the early COVID pandemic situation in Australia. The reliability of the above proposed model is clearly visualized by figure 1 where growing spread and declining spread are considered separately by an exponential model of order 2. The influences of different mitigation strategies implemented by Australia can be revealed by closely examining on $f_i(t)s$ depicted on figure 2.

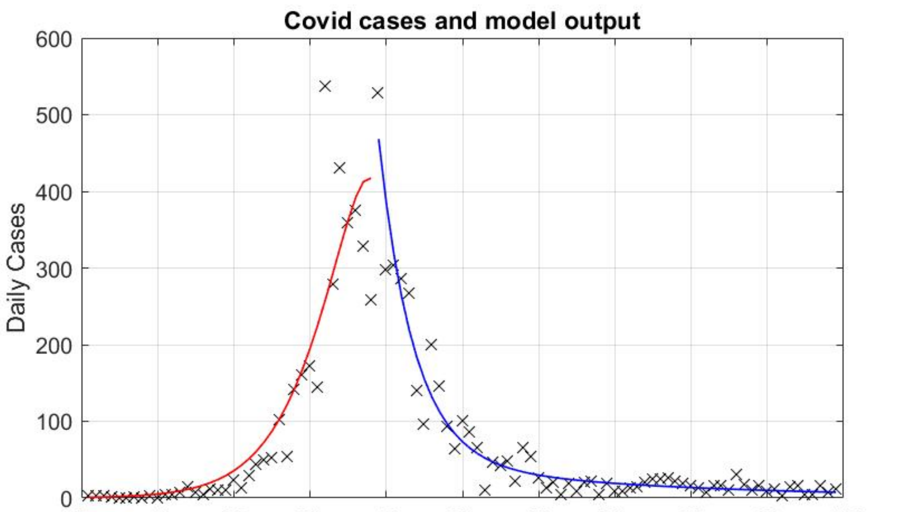

Figure 1: The model applied for growing spread and declining spread of active cases of COVID-19 in Australia

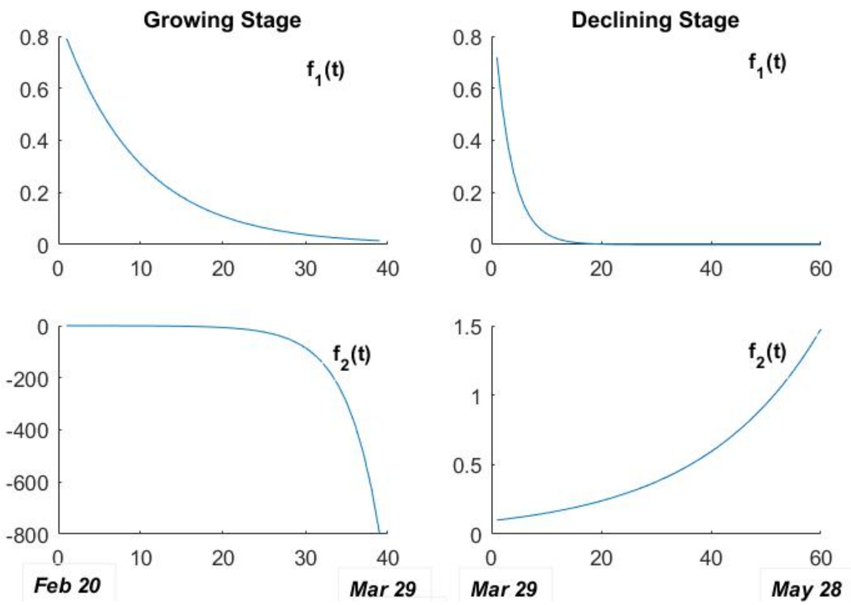

Figure 2: Competency functions of each basis function in inclining and declining stages

2.2 Constant Interval Integration

In the recent outbreak of COVID-19, infectious period is identified as a important factor, in which a proper controlling strategy could diminish the spread. For every infective, the infectious period is a constant interval irrespective of when they are exposed to the disease. Thus in terms of epidemics, following delayed-Volterra integral equation can be interpreted as a model for the spread of certain infectious diseases.

$$x(t) = \int_{t-L}^{t} g(t, s, x(s))ds \tag{4}$$

Assuming $x(t)$ represents the proportion of infected in population at the time t, $g(t, s, x(s))$ represents the proportion of new infected in population per unit time and L is the duration of infectious period. Accordingly, the number of infectious individuals at time t is equal to the sum of all individuals infected between $t - L$ and t. In the early stage of COVID-19 pandemic before a community spread, infected cases are recognized in cluster wise and the controlling strategies are laid cluster wise. In consequence, the model (5) indicates a potential in introducing this cluster wise transmission. In the parsimonious approach $g(t, s, x(s))$ can be defined as $g(t, s, x(s)) = f(t - s)x(s)$ where $f(t)$ illustrates the competency of spreading.

$$\begin{aligned} x(t) &= \int_{t-L}^{t} f(t-s)x(s)ds \text{ where } t \in [0,t] \\ x(t) &= \psi(t) \text{ where } t \in [-L,0] \end{aligned} \tag{5}$$

Primarily one could consider the impact of quarantine effect, government regulations, health care facility, mobility records to proper quantification of cluster wise transmission.

2.3 Fixed Interval Integration

The fixed interval integration concept widens the applicability of describing different perspectives of disease transmission such as spatial distribution, age-specific distribution, etc. It will pave the way to introduce factors in different platforms which shows vulnerability as an amalgamated effect. Such factors can be elucidate as follows by using variable separable functions $(f(t), g(x))$ if they can be expressed as the sum of a finite number of terms, each of which is the product of functions of two platforms [7].

$$K(t,a) = \sum_{i=1}^{n} f_i(t) g_i(a) ds \tag{6}$$

In COVID-19 transmission also, we observed the mortality and morbidity depends on chronological age as the youngest and the eldest are having a great deal of attention [8]. The standard formalism of age-structured differential models is to subdivide the population into a number of discrete compartments, classified by the age groups which is appeared as a system of differential equations [4]. This compartmental approach seems more realistic when only the age range within each compartment is priorly determined.

In contrast to that, integral equations can consider chronological age as a continuous variable that enables to formulate the risk of being infectious $(u(a))$ in different ages as follows,

$$u(a) - h(a) = \lambda \int_{0}^{L} K(a,t)u(t)dt \tag{7}$$

Here $h(a)$ can be used as the biological ability to respond the disease and $K(a,t)$ introduce the amalgamated effect by the behavioral influence $(g(a))$ which depends on age and environmental impact $(f(t))$ which depends on time.

3 Discussion

It is possible to model most of the general phenomena in COVID-19 transmission via integral equations. Since lockdown and quarantine interval shows a control in the spread, it is evident that the transmission can be formulated as an accumulated impact of certain factors in that period in which concerning of rate of change would be rather difficult. These contribution of accumulated impacts can be tenderly managed in their governing period. Thus integral equation formulation is more elegant than differential equations, as fixed intervals, constant intervals and variable intervals can be easily considered. Further in the COVID context as there exist a continuously observed dataset, integral equation modeling is not cumbersome.

In spite of that, the kernel, main modelling entity of integral equations paves a way to address different behaviours in transmission. Degenerated kernel could address when factors in different platforms evoke as an amalgamated effect.Thus both age and time or spatial and time related influences can be easily formulated. In the point of view of accumulation behaviors, there are some cases where the causative factors may not affect immediately, but with a time-lapse as time lag is a general phenomenon in epidemiology. In the COVID pandemic, we observed the similar serious consequences due to the time delay in awareness campaigns and policy decisions.

Conversion of IVP to Volterra integral indicates a significant advantage in modelling in terms of integrals, as Volterra integrals with difference kernel converts to a second order differential equation. Thus modeling a real world phenomena in terms of second order differential equation is burdensome. Not only in the modelling perspective, but also in the solving perspective, integral representations are more appropriate when there is no closed form solutions. One of the reasons is that the truncation error of the numerical solution of integral equation tend to be averaged by the process of quadrature while the error tends to accumulate in the process of numerical differentiation[9].

This is not a comparative study of modelling COVID pandemic using differential models and integral models, but aims on disclosing the applicability of integral equation models in the above context.

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