Journal Homepage: - www.journalijar.com
INTERNATIONAL JOURNAL OF ADVANCED RESEARCH (IJAR)
Article DOI: 10.21474/IJAR01/5043
DOI URL: http://dx.doi.org/10.21474/IJAR01/5043

***RESEARCH ARTICLE***

**EXPLICIT FORMULAS FOR THE EXPONENTIALS OF SOME SPECIAL MATRICES.**

**K. K. W. A. S Kumara and W. L. C. Piyasundara.**
University of Sri Jayewardenepura, Gangodawila, Nugegoda, Sri Lanka.

***Manuscript Info***

*Manuscript History*

Received: 02 June 2017
Final Accepted: 04 July 2017
Published: August 2017

***Key words:-***
Exponential, Matrices, power series

***Abstract***

The matrix exponential has many applications in the fields of mathematics, physics and economics. There are many explicit formulas that have been developed for compute the matrix exponential. In this paper we give some explicit formulas for the exponentials of some special matrices. The main results are the extension of Beibei Wu's work.



## Introduction:-

Consider the linear vector differential equation $\dot{x}(t) = Ax(t)$, where $x(t)$ is an n-vector and A is an $n \times n$ matrix. It plays a fundamental role in the study of dynamical systems and linear control systems. It is well known that the solution to this equation is given by $x(t) = e^{At}x_0$, where $e^{At}$ denotes the exponential of the matrix A times t and can be identified as the convergent power series

$$e^{At} = \sum_{i=0}^{\infty} \frac{(At)^i}{i!}.$$

Therefore, it is important to have accurate numerical methods for computing the matrix exponential function. As a result of this, many explicit formulas have been developed for the matrix exponential by many authors. Through this work, we also hope to give explicit formulas for computing the exponentials of some special matrices.

## Main Results:-

Denote the set of non-negative integers by $N_0$, the set of complex numbers by C, and the set of all $n \times n$ complex matrices by $C^{n\times n}$.The symbols $O_n$ *and* $I_n$ will be used to denote the $n \times n$ zero matrix and the $n \times n$ identity matrix, respectively.

Bernstein and So gave explicit formulae for $A^2 = A$, $A^2 = \rho I_n$ and $A^3 = \rho A, \rho \in C$ and Beibei Wu gave explicit formulae for $A^{k+1} = \rho A^k$, $A^{k+2} = \rho^2 A^k$ and $A^{k+3} = \rho^3 A^k$, $\rho \in C$ and $k \in N_0$. Now we hope to extend Beibei Wu's results to the general cases. Furthermore, we derive explicit formulae for computing the exponentials of some special matrices that satisfy polynomials $A^{k+4r} = \rho^{4r} A^k$ and $A^{k+(4r+2)} = \rho^{(4r+2)} A^k$, $\rho \in C;\ k, r \in N_0$.

**Corresponding Author:- W. L. C. Piyasundara.**
Address:- University of Sri Jayewardenepura, Gangodawila, Nugegoda, Sri Lanka.

**Theorem 1.** Let $\in C^{n\times n}$ , where $A^{k+4r} = \rho^{4r}A^k, \rho \in C;\ k,r \in N_0$.

i. If $\rho = 0$ then

$$e^A = \sum_{i=0}^{k+4r-1} \frac{A^i}{i!}$$

ii. If $\rho \neq 0, k = (4r)l\ (l \in N_0)$ ,then

$$e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!}$$

$$= \sum_{i=0}^{k-1} \frac{A^i}{i!} + \left(\sum_{m=k/4r}^{\infty} \frac{\rho^{(4r)m-k}}{(4rm)!}\right)A^k + \left(\sum_{m=k/4r}^{\infty} \frac{\rho^{(4r)m-k}}{(4rm+1)!}\right)A^{k+1} + \left(\sum_{m=k/4r}^{\infty} \frac{\rho^{(4r)m-k}}{(4rm+2)!}\right)A^{k+2} + \cdots + \left(\sum_{m=k/4r}^{\infty} \frac{\rho^{(4r)m-k}}{\left(4rm+(4r-1)\right)!}\right)A^{k+(4r-1)}$$

$$= \sum_{i=0}^{k-1} \frac{A^i}{i!} + \frac{1}{\rho^k}\left(\sum_{m=0}^{\infty} \frac{\rho^{4rm}}{(4rm)!} - \sum_{m=0}^{(k/4r)-1} \frac{\rho^{4rm}}{(4rm)!}\right)A^k + \frac{1}{\rho^{k+1}}\left(\sum_{m=0}^{\infty} \frac{\rho^{4rm+1}}{(4rm+1)!} - \sum_{m=0}^{(k/4r)-1} \frac{\rho^{4rm+1}}{(4rm+1)!}\right)A^{k+1} + \frac{1}{\rho^{k+2}}\left(\sum_{m=0}^{\infty} \frac{\rho^{4rm+2}}{(4rm+2)!} - \sum_{m=0}^{(k/4r)-1} \frac{\rho^{4rm+2}}{(4rm+2)!}\right)A^{k+2} + \cdots + \frac{1}{\rho^{k+(4r-1)}}\left(\sum_{m=0}^{\infty} \frac{\rho^{4rm+(4r-1)}}{\left(4rm+(4r-1)\right)!} - \sum_{m=0}^{(k/4r)-1} \frac{\rho^{4rm+(4r-1)}}{\left(4rm+(4r-1)\right)!}\right)A^{k+(4r-1)},$$

where

$$\sum_{m=0}^{\infty} \frac{\rho^{4rm}}{(4rm)!} = \frac{1}{\left(1+\frac{1}{\cos(\pi/4r)}\right)}\left[e^{\rho} - \frac{1}{\cos(\pi/4r)} e^{-\rho\cos\left(\frac{\pi}{4r}\right)} \cos\left(\rho \sin\left(\frac{\pi}{4r}\right)\right)\right]; r = 1,2,3, \ldots$$

**Proof:**

$$e^{\rho} - \frac{1}{\cos(\pi/4r)} e^{-\rho\cos\left(\frac{\pi}{4r}\right)} \cos\left(\rho \sin\left(\frac{\pi}{4r}\right)\right)$$

$$= e^{\rho} - \frac{1}{\cos(\pi/4r)} e^{-\rho\cos\left(\frac{\pi}{4r}\right)} \left(\frac{e^{i\rho\sin\left(\frac{\pi}{4r}\right)} + e^{-i\rho\sin\left(\frac{\pi}{4r}\right)}}{2}\right)$$

$$= e^{\rho} - \frac{1}{\cos(\pi/4r)}\left[\frac{1}{2} e^{\left(\cos\left(\pi-\frac{\pi}{4r}\right) + i\sin\left(\pi-\frac{\pi}{4r}\right)\right)\rho} + \frac{1}{2} e^{\left(\cos\left(\pi+\frac{\pi}{4r}\right) + i\sin\left(\pi+\frac{\pi}{4r}\right)\right)\rho}\right]$$

$$= e^{\rho} - \frac{1}{\cos(\pi/4r)}\left[\frac{1}{2}e^{\left(\cos\frac{(4r-1)\pi}{4r} + i\sin\frac{(4r-1)\pi}{4r}\right)\rho} + \frac{1}{2}e^{\left(\cos\frac{(4r+1)\pi}{4r} + i\sin\frac{(4r+1)\pi}{4r}\right)\rho}\right]$$

$$= \sum_{n=0}^{\infty}\frac{\rho^n}{n!} - \frac{1}{\cos(\pi/4r)}\left[\frac{1}{2}\sum_{n=0}^{\infty}\frac{\left(\cos\frac{(4r-1)\pi}{4r} + i\sin\frac{(4r-1)\pi}{4r}\right)^n\rho^n}{n!} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{\left(\cos\frac{(4r+1)\pi}{4r} + i\sin\frac{(4r+1)\pi}{4r}\right)^n\rho^n}{n!}\right]$$

By De Moivre's theorem,

$$\left(\cos\frac{(4r-1)\pi}{4r} + i\sin\frac{(4r-1)\pi}{4r}\right)^n = \cos\frac{(4r-1)n\pi}{4r} + i\sin\frac{(4r-1)n\pi}{4r} \quad \text{and}$$
$$\left(\cos\frac{(4r+1)\pi}{4r} + i\sin\frac{(4r+1)\pi}{4r}\right)^n = \cos\frac{(4r+1)n\pi}{4r} + i\sin\frac{(4r+1)n\pi}{4r}.$$

Hence we get,

$$e^{\rho} - \frac{1}{\cos(\pi/4r)}e^{-\rho\cos\left(\frac{\pi}{4r}\right)}\cos\left(\rho\sin\left(\frac{\pi}{4r}\right)\right)$$

$$= \sum_{n=0}^{\infty}\frac{\rho^n}{n!}\left\{1 - \frac{1}{2\cos(\pi/4r)}\left[\cos\frac{(4r-1)n\pi}{4r} + i\sin\frac{(4r-1)n\pi}{4r} + \cos\frac{(4r+1)n\pi}{4r} + i\sin\frac{(4r+1)n\pi}{4r}\right]\right\}$$

$$= \sum_{n=0}^{\infty}\frac{\rho^n}{n!}\left\{1 - \frac{1}{2\cos(\pi/4r)}\left[\left(\cos\frac{(4r-1)n\pi}{4r} + \cos\frac{(4r+1)n\pi}{4r}\right) + i\left(\sin\frac{(4r-1)n\pi}{4r} + \sin\frac{(4r+1)n\pi}{4r}\right)\right]\right\}$$

$$= \sum_{n=0}^{\infty}\frac{\rho^n}{n!}\left[1 - \frac{1}{2\cos(\pi/4r)}\left(2\cos n\pi\cos\frac{n\pi}{4r}\right) - \frac{i}{2\cos(\pi/4r)}\left(2\sin n\pi\cos\frac{n\pi}{4r}\right)\right]$$

Since, $\cos n\pi = (-1)^n$ and $\sin n\pi = 0$ ,

$$e^{\rho} - \frac{1}{\cos(\pi/4r)}e^{-\rho\cos\left(\frac{\pi}{4r}\right)}\cos\left(\rho\sin\left(\frac{\pi}{4r}\right)\right) = \sum_{n=0}^{\infty}\frac{\rho^n}{n!}\left[1 - \frac{1}{\cos(\pi/4r)}(-1)^n\cos\frac{n\pi}{4r}\right]$$

$$= \sum_{m=0}^{\infty}\frac{\rho^{4rm}}{(4rm)!}\left[1 - \frac{1}{\cos(\pi/4r)}(-1)^{4rm}\cos\frac{4rm\pi}{4r}\right]$$
$$+ \sum_{m=0}^{\infty}\frac{\rho^{4rm+1}}{(4rm+1)!}\left[1 - \frac{1}{\cos(\pi/4r)}(-1)^{4rm+1}\cos\frac{(4rm+1)\pi}{4r}\right]$$
$$+ \sum_{m=0}^{\infty}\frac{\rho^{4rm+2}}{(4rm+2)!}\left[1 - \frac{1}{\cos(\pi/4r)}(-1)^{4rm+2}\cos\frac{(4rm+2)\pi}{4r}\right] + \cdots$$
$$+ \sum_{m=0}^{\infty}\frac{\rho^{4rm+(4r-1)}}{(4rm+4r-1)!}\left[1 - \frac{1}{\cos(\pi/4r)}(-1)^{4rm+(4r-1)}\cos\frac{\left(4rm+(4r-1)\right)\pi}{4r}\right].$$

$$= \sum_{m=0}^{\infty} \frac{\rho^{4rm}}{(4rm)!}\left[1 - \frac{1}{\cos(\pi/4r)}(-1)^{4\mathrm{rm}}(-1)^{\mathrm{m}}\right] + A + B + \cdots + C\,,$$

where

$$A = \sum_{m=0}^{\infty} \frac{\rho^{4rm+1}}{(4rm+1)!}\left[1 - \frac{1}{\cos(\pi/4r)}(-1)^{4\mathrm{rm}+1}\cos\frac{(4\mathrm{rm}+1)\pi}{4\mathrm{r}}\right],$$

$$B = \sum_{m=0}^{\infty} \frac{\rho^{4rm+2}}{(4rm+2)!}\left[1 - \frac{1}{\cos(\pi/4r)}(-1)^{4\mathrm{rm}+2}\cos\frac{(4\mathrm{rm}+2)\pi}{4\mathrm{r}}\right] \text{ and}$$

$$C = \sum_{m=0}^{\infty} \frac{\rho^{4rm+(4r-1)}}{(4rm+4r-1)!}\left[1 - \frac{1}{\cos(\pi/4r)}(-1)^{4rm+(4r-1)}\cos\frac{\big(4rm+(4r-1)\big)\pi}{4\mathrm{r}}\right].$$

Now let $\cos\frac{(4\mathrm{rm}+1)\pi}{4\mathrm{r}} = \beta$.

Then $\cos 4\mathrm{r}\left(\frac{(4\mathrm{rm}+1)\pi}{4\mathrm{r}}\right) = \cos\big((4\mathrm{rm}+1)\pi\big) = (-1)^{4\mathrm{rm}+1} = -(-1)^{4\mathrm{rm}}$.

Let $\beta = -\cos\left(\frac{\pi}{4r}\right)(-1)^{4\mathrm{rm}}$

It implies that, $\cos\left(\frac{(4\mathrm{rm}+1)\pi}{4\mathrm{r}}\right) = -\cos\left(\frac{\pi}{4r}\right)(-1)^{4\mathrm{rm}}$.

Now

$$A = \sum_{m=0}^{\infty} \frac{\rho^{4rm+1}}{(4rm+1)!}\left[1 - \frac{1}{\cos(\pi/4r)}(-1)^{4\mathrm{rm}+1}\cos\frac{(4\mathrm{rm}+1)\pi}{4\mathrm{r}}\right].$$

$$= \sum_{m=0}^{\infty} \frac{\rho^{4rm+1}}{(4rm+1)!}\left[1 + \frac{1}{\cos(\pi/4r)}(-1)^{4\mathrm{rm}}\left(-\cos\left(\frac{\pi}{4r}\right)(-1)^{4\mathrm{rm}}\right)\right] = \sum_{m=0}^{\infty} \frac{\rho^{4rm+1}}{(4rm+1)!}[1-1]$$

$$= 0$$

Similarly,

$$\cos\left(\frac{(4\mathrm{rm}+2)\pi}{4\mathrm{r}}\right) = -\cos\left(\frac{\pi}{4r}\right)(-1)^{4\mathrm{rm}+1} \text{ and hence}$$

$$B = \sum_{m=0}^{\infty} \frac{\rho^{4rm+2}}{(4rm+2)!}\left[1 - \frac{1}{\cos\left(\frac{\pi}{4r}\right)}(-1)^{4\mathrm{rm}+2}\cos\frac{(4\mathrm{rm}+2)\pi}{4\mathrm{r}}\right]$$

$$= \sum_{m=0}^{\infty} \frac{\rho^{4rm+2}}{(4rm+2)!}\left[1 - \frac{1}{\cos(\pi/4r)}(-1)^{4\mathrm{rm}}\left(-\cos\left(\frac{\pi}{4r}\right)(-1)^{4\mathrm{rm}+1}\right)\right] = 0\text{, and}$$

$$\cos\frac{\big(4rm+(4r-1)\big)\pi}{4\mathrm{r}} = -\cos\left(\frac{\pi}{4r}\right)(-1)^{4\mathrm{rm}+4\mathrm{r}-2} \text{ and hence}$$

$$C = \sum_{m=0}^{\infty} \frac{\rho^{4rm+(4r-1)}}{(4rm+4r-1)!}\left[1 - \frac{1}{\cos(\pi/4r)}(-1)^{4rm+(4r-1)}\cos\frac{\big(4rm+(4r-1)\big)\pi}{4\mathrm{r}}\right] = 0.$$

Therefore,

$$e^{\rho} - \frac{1}{\cos(\pi/4r)} e^{-\rho\cos\left(\frac{\pi}{4r}\right)} \cos\left(\rho \sin\left(\frac{\pi}{4r}\right)\right) = \sum_{m=0}^{\infty} \frac{\rho^{4rm}}{(4rm)!}\left[1 - \frac{1}{\cos(\pi/4r)}(-1)^{(4r+1)m}\right]$$

$$= \sum_{m=0}^{\infty} \frac{\rho^{4rm}}{(4rm)!}\left[1 + \frac{1}{\cos(\pi/4r)}\right]$$

Hence, we get

$$\sum_{m=0}^{\infty} \frac{\rho^{4rm}}{(4rm)!} = \frac{1}{\left(1 + \frac{1}{\cos(\pi/4r)}\right)}\left[e^{\rho} - \frac{1}{\cos(\pi/4r)} e^{-\rho\cos\left(\frac{\pi}{4r}\right)} \cos\left(\rho \sin\left(\frac{\pi}{4r}\right)\right)\right]; r = 1,2,3, \ldots \blacksquare$$

By integrating this result by $\rho$ we can obtain the formulas for remaining infinite power series.

Similarly we can derive the formulas for $e^A$ when

$k = (4r)l + 1, (4r)l + 2, \ldots, (4r)l + (4r - 1);\ (l \in N_0)$.

Next, we consider the case in which A satisfies that $A^{k+(4r+2)} = \rho^{(4r+2)} A^k$, $\rho \in C;\ k, r \in N_0$.

**Theorem 2.** Let $\in C^{n\times n}$ , where $A^{k+(4r+2)} = \rho^{(4r+2)} A^k$, $\rho \in C;\ k, r \in N_0$.

i. If $\rho = 0$ then

$$e^A = \sum_{i=0}^{k+4r+1} \frac{A^i}{i!}$$

ii. If $\rho \neq 0, k = (4r+2)l\ (l \in N_0)$ ,then

$$e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!}$$

$$= \sum_{i=0}^{k-1} \frac{A^i}{i!} + \left(\sum_{m=k/(4r+2)}^{\infty} \frac{\rho^{(4r+2)m-k}}{((4r+2)m)!}\right) A^k + \left(\sum_{m=k/(4r+2)}^{\infty} \frac{\rho^{(4r+2)m-k}}{((4r+2)m+1)!}\right) A^{k+1}$$
$$+ \left(\sum_{m=k/(4r+2)}^{\infty} \frac{\rho^{(4r+2)m-k}}{((4r+2)m+2)!}\right) A^{k+2} + \cdots + \left(\sum_{m=k/(4r+2)}^{\infty} \frac{\rho^{(4r+2)m-k}}{((4r+2)m+(4r+1))!}\right) A^{k+(4r+1)}$$

$$= \sum_{i=0}^{k-1} \frac{A^i}{i!} + \frac{1}{\rho^k}\left(\sum_{m=0}^{\infty} \frac{\rho^{(4r+2)m}}{((4r+2)m)!} - \sum_{m=0}^{(k/(4r+2))-1} \frac{\rho^{(4r+2)m}}{((4r+2)m)!}\right) A^k$$
$$+ \frac{1}{\rho^{k+1}}\left(\sum_{m=0}^{\infty} \frac{\rho^{(4r+2)m+1}}{((4r+2)m+1)!} - \sum_{m=0}^{(k/(4r+2))-1} \frac{\rho^{(4r+2)m+1}}{((4r+2)m+1)!}\right) A^{k+1}$$
$$+ \frac{1}{\rho^{k+2}}\left(\sum_{m=0}^{\infty} \frac{\rho^{(4r+2)m+2}}{((4r+2)m+2)!} - \sum_{m=0}^{(k/(4r+2))-1} \frac{\rho^{(4r+2)m+2}}{((4r+2)m+2)!}\right) A^{k+2} + \cdots$$

$$+\frac{1}{\rho^{k+(4r+1)}}\left(\sum_{m=0}^{\infty}\frac{\rho^{(4r+2)m+(4r+1)}}{\big((4r+2)m+(4r+1)\big)!}-\sum_{m=0}^{(k/(4r+2))-1}\frac{\rho^{(4r+2)m+(4r+1)}}{\big((4r+2)m+(4r+1)\big)!}\right)A^{k+(4r+1)}$$

where,

$$\sum_{m=0}^{\infty}\frac{\rho^{(4r+2)m}}{\big((4r+2)m\big)!}$$
$$=\frac{1}{\left(1-\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}\right)}\left[e^{\rho}+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}e^{-\rho\cos\left(\frac{\pi}{4r+2}\right)}\cos\left(\rho\sin\left(\frac{\pi}{4r+2}\right)\right)\right];r=1,2,3,\ldots$$

**Proof:**

$$e^{\rho}+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}e^{-\rho\cos\left(\frac{\pi}{4r+2}\right)}\cos\left(\rho\sin\left(\frac{\pi}{4r+2}\right)\right)$$

$$=e^{\rho}+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}e^{-\rho\cos\left(\frac{\pi}{4r+2}\right)}\left(\frac{e^{i\rho\sin\left(\frac{\pi}{4r+2}\right)}+e^{-i\rho\sin\left(\frac{\pi}{4r+2}\right)}}{2}\right)$$

$$=e^{\rho}+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}\left[\frac{1}{2}e^{\left(\cos\left(\pi-\frac{\pi}{4r+2}\right)+i\sin\left(\pi-\frac{\pi}{4r+2}\right)\right)\rho}+\frac{1}{2}e^{\left(\cos\left(\pi+\frac{\pi}{4r+2}\right)+i\sin\left(\pi+\frac{\pi}{4r+2}\right)\right)\rho}\right]$$

$$=e^{\rho}+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}\left[\frac{1}{2}e^{\left(\cos\frac{(4r+1)\pi}{4r+2}+i\sin\frac{(4r+1)\pi}{4r+2}\right)\rho}+\frac{1}{2}e^{\left(\cos\frac{(4r+3)\pi}{4r+2}+i\sin\frac{(4r+3)\pi}{4r+2}\right)\rho}\right]$$

$$=\sum_{n=0}^{\infty}\frac{\rho^n}{n!}+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}\left[\frac{1}{2}\sum_{n=0}^{\infty}\frac{\left(\cos\frac{(4r+1)\pi}{4r+2}+i\sin\frac{(4r+1)\pi}{4r+2}\right)^n\rho^n}{n!}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{\left(\cos\frac{(4r+3)\pi}{4r+2}+i\sin\frac{(4r+3)\pi}{4r+2}\right)^n\rho^n}{n!}\right]$$

By De Moivre’s theorem,

$$\left(\cos\frac{(4r+1)\pi}{4r+2}+i\sin\frac{(4r+1)\pi}{4r+2}\right)^n=\cos\frac{(4r+1)n\pi}{4r+2}+i\sin\frac{(4r+1)n\pi}{4r+2}\quad\text{and}$$

$$\left(\cos\frac{(4r+3)\pi}{4r+2}+i\sin\frac{(4r+3)\pi}{4r+2}\right)^n=\cos\frac{(4r+3)n\pi}{4r+2}+i\sin\frac{(4r+3)n\pi}{4r+2}.$$

Hence we get,

$$e^{\rho}+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}e^{-\rho\cos\left(\frac{\pi}{4r+2}\right)}\cos\left(\rho\sin\left(\frac{\pi}{4r+2}\right)\right)$$

$$=\sum_{n=0}^{\infty}\frac{\rho^n}{n!}\left\{1+\frac{1}{2\cos\left(\frac{\pi}{4r+2}\right)}\left[\cos\frac{(4\mathrm{r}+1)\mathrm{n}\pi}{4\mathrm{r}+2}+\mathrm{i}\sin\frac{(4\mathrm{r}+1)\mathrm{n}\pi}{4\mathrm{r}+2}+\cos\frac{(4\mathrm{r}+3)\mathrm{n}\pi}{4\mathrm{r}+2}+\mathrm{i}\sin\frac{(4\mathrm{r}+3)\mathrm{n}\pi}{4\mathrm{r}+2}\right]\right\}$$

$$=\sum_{n=0}^{\infty}\frac{\rho^n}{n!}\left\{1+\frac{1}{2\cos\left(\frac{\pi}{4r+2}\right)}\left[\left(\cos\frac{(4\mathrm{r}+1)\mathrm{n}\pi}{4\mathrm{r}+2}+\cos\frac{(4\mathrm{r}+3)\mathrm{n}\pi}{4\mathrm{r}+2}\right)+\mathrm{i}\left(\sin\frac{(4\mathrm{r}+1)\mathrm{n}\pi}{4\mathrm{r}+2}+\sin\frac{(4\mathrm{r}+3)\mathrm{n}\pi}{4\mathrm{r}+2}\right)\right]\right\}$$

$$=\sum_{n=0}^{\infty}\frac{\rho^n}{n!}\left[1+\frac{1}{2\cos\left(\frac{\pi}{4r+2}\right)}\left(2\cos\,\mathrm{n}\pi\cos\frac{\mathrm{n}\pi}{4\mathrm{r}+2}\right)+\frac{i}{2\cos\left(\frac{\pi}{4r+2}\right)}\left(2\sin\mathrm{n}\pi\cos\frac{\mathrm{n}\pi}{4\mathrm{r}+2}\right)\right]$$

Since, $\cos\ \mathrm{n}\pi=(-1)^{\mathrm{n}}$ and $\sin\mathrm{n}\pi=0$ ,

$$e^{\rho}+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}e^{-\rho\cos\left(\frac{\pi}{4r+2}\right)}\cos\left(\rho\sin\left(\frac{\pi}{4r+2}\right)\right)$$

$$=\sum_{n=0}^{\infty}\frac{\rho^n}{n!}\left[1+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}(-1)^{\mathrm{n}}\cos\frac{\mathrm{n}\pi}{4\mathrm{r}+2}\right]$$

$$=\sum_{m=0}^{\infty}\frac{\rho^{(4r+210)m}}{\left((4r+2)m\right)!}\left[1+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}(-1)^{(4r+2)\mathrm{m}}\cos\frac{(4r+2)\mathrm{m}\pi}{(4r+2)}\right]$$
$$+\sum_{m=0}^{\infty}\frac{\rho^{(4r+2)m+1}}{\left((4r+2)m+1\right)!}\left[1+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}(-1)^{(4r+2)\mathrm{m}+1}\cos\frac{\left((4r+2)\mathrm{m}+1\right)\pi}{(4r+2)}\right]$$
$$+\sum_{m=0}^{\infty}\frac{\rho^{(4r+2)m+2}}{\left((4r+2)m+2\right)!}\left[1+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}(-1)^{(4r+2)\mathrm{m}+2}\cos\frac{\left((4r+2)\mathrm{m}+2\right)\pi}{(4r+2)}\right]+\cdots$$
$$+\sum_{m=0}^{\infty}\frac{\rho^{(4r+2)m+(4r-1)}}{\left((4r+2)m+(4r+2)-1\right)!}\left[1+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}(-1)^{(4r+2)m+\left((4r+2)-1\right)}\cos\frac{\left((4r+2)m+\left((4r+2)-1\right)\right)\pi}{(4r+2)}\right].$$

$$=\sum_{m=0}^{\infty}\frac{\rho^{(4r+2)m}}{\left((4r+2)m\right)!}\left[1+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}(-1)^{(4r+2)\mathrm{m}}(-1)^{\mathrm{m}}\right]+A+B+\cdots+C\ ,$$

where

$$A=\sum_{m=0}^{\infty}\frac{\rho^{(4r+2)m+1}}{\left((4r+2)m+1\right)!}\left[1+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}(-1)^{(4r+2)\mathrm{m}+1}\cos\frac{\left((4r+2)\mathrm{m}+1\right)\pi}{(4r+2)}\right],$$
$$B=\sum_{m=0}^{\infty}\frac{\rho^{(4r+2)m+2}}{\left((4r+2)m+2\right)!}\left[1+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}(-1)^{(4r+2)\mathrm{m}+2}\cos\frac{\left((4r+2)\mathrm{m}+2\right)\pi}{(4r+2)}\right]\ \text{and}$$

$$C = \sum_{m=0}^{\infty} \frac{\rho^{(4r+2)m+(4r-1)}}{\big((4r+2)m+(4r+2)-1\big)!}\left[1+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}(-1)^{(4r+2)m+((4r+2)-1)}\cos\frac{\big((4r+2)m+(4r+1)\big)\pi}{(4r+2)}\right].$$

Now let $\cos\frac{((4r+2)\mathrm{m}+1)\pi}{(4r+2)} = \beta$.

Then $\cos(4r+2)\left(\frac{((4r+2)\mathrm{m}+1)\pi}{(4r+2)}\right) = \cos\big(((4r+2)\mathrm{m}+1)\pi\big) = (-1)^{(4r+2)\mathrm{m}+1} = -(-1)^{(4r+2)\mathrm{m}}$.

Let $\beta = \cos\left(\frac{\pi}{4r+2}\right)(-1)^{(4r+2)\mathrm{m}}$

It implies that, $\cos\frac{\big((4r+2)\mathrm{m}+1\big)\pi}{(4r+2)} = \cos\left(\frac{\pi}{4r+2}\right)(-1)^{(4r+2)\mathrm{m}}$ .

Now

$$A = \sum_{m=0}^{\infty} \frac{\rho^{(4r+2)m+1}}{\big((4r+2)m+1\big)!}\left[1+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}(-1)^{(4r+2)\mathrm{m}+1}\cos\frac{\big((4r+2)\mathrm{m}+1\big)\pi}{(4r+2)}\right].$$

$$= \sum_{m=0}^{\infty} \frac{\rho^{(4r+2)m+1}}{\big((4r+2)m+1\big)!}\left[1-\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}(-1)^{(4r+2)\mathrm{m}}\left(\cos\left(\frac{\pi}{4r+2}\right)(-1)^{(4r+2)\mathrm{m}}\right)\right]$$

$$= \sum_{m=0}^{\infty} \frac{\rho^{(4r+2)m+1}}{\big((4r+2)m+1\big)!}[1-1] = 0$$

Similarly,

$\cos\frac{\big((4r+2)\mathrm{m}+2\big)\pi}{(4r+2)} = \cos\left(\frac{\pi}{4r+2}\right)(-1)^{(4r+2)\mathrm{m}+1}$ and hence

$$B = \sum_{m=0}^{\infty} \frac{\rho^{(4r+2)m+2}}{\big((4r+2)m+2\big)!}\left[1+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}(-1)^{(4r+2)\mathrm{m}+2}\cos\frac{\big((4r+2)\mathrm{m}+2\big)\pi}{(4r+2)}\right]$$

$$= \sum_{m=0}^{\infty} \frac{\rho^{(4r+2)m+2}}{\big((4r+2)m+2\big)!}\left[1+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}(-1)^{(4r+2)\mathrm{m}+2}\left(\cos\left(\frac{\pi}{4r+2}\right)(-1)^{(4r+2)\mathrm{m}+1}\right)\right]$$

$= 0$ , and

$\cos\frac{\big((4r+2)m+(4r+1)\big)\pi}{(4r+2)} = \cos\left(\frac{\pi}{4r+2}\right)(-1)^{(4r+2)\mathrm{m}+4r}$ and hence

$$C = \sum_{m=0}^{\infty} \frac{\rho^{(4r+2)m+(4r-1)}}{\big((4r+2)m+(4r+2)-1\big)!}\left[1+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}(-1)^{(4r+2)m+((4r+2)-1)}\cos\frac{\big((4r+2)m+(4r+1)\big)\pi}{(4r+2)}\right]$$

$= 0$.

Therefore,

$$e^{\rho}+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}e^{-\rho\cos\left(\frac{\pi}{4r+2}\right)}\cos\left(\rho\sin\left(\frac{\pi}{4r+2}\right)\right)$$

$$=\sum_{m=0}^{\infty}\frac{\rho^{(4r+2)m}}{\left((4r+2)m\right)!}\left[1+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}(-1)^{(4r+3)\mathrm{m}}\right]$$

$$=\sum_{m=0}^{\infty}\frac{\rho^{(4r+2)m}}{\left((4r+2)m\right)!}\left[1-\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}\right]$$

Hence, we get

$$\sum_{m=0}^{\infty}\frac{\rho^{(4r+2)m}}{\left((4r+2)m\right)!}$$
$$=\frac{1}{\left(1-\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}\right)}\left[e^{\rho}+\frac{1}{\cos\left(\frac{\pi}{4r+2}\right)}e^{-\rho\cos\left(\frac{\pi}{4r+2}\right)}\cos\left(\rho\sin\left(\frac{\pi}{4r+2}\right)\right)\right];r=1,2,3,\ldots\blacksquare$$

By integrating this result with respect to $\rho$ we can obtain the formulas for remaining infinite power series.

Similarly we can derive the formulas for $e^{A}$ when

$k=(4r+2)l+1,(4r+2)l+2,\ldots,(4r+2)l+(4r+1);\ (l\in N_0)$.

## References:-

1. B.Wu, 'Explicit formulas for the exponentials of some special matrices', Shanghai University of Electric Power, Applied Mathematics Letters 24 (2011), pp. 642-647.
2. C.Moler,C.Loan, 'Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty- Five Years Later', SIAM Review 45 ,2003 , pp. 3-49.
3. D. Bernstein, S. Wasin, 'Some explicit formulas for the matrix exponential', vol.38 , no.8 , 1993.
4. G.Walz, 'Computing the matrix exponential and other matrix functions' , 1988.
5. Nn.D.Higham,'How and how not to compute the exponential of a matrix', The University of Manchester.
6. T.Apostol, 'Some explicit formulas for the matrix exponential, Amer. Math', 1969 , pp.284- 292.